



**You have downloaded a document from  
RE-BUS  
repository of the University of Silesia in Katowice**

**Title:** Cycles in Nielsen's graphs

**Author:** Anna Bień

**Citation style:** Bień Anna. (2010). Cycles in Nielsen's graphs. "Bulletin of the Section of Logic" (Vol. 39 no. 1/2 (2010), s. 57-64).



Uznanie autorstwa - Użycie niekomercyjne - Bez utworów zależnych Polska - Licencja ta zezwala na rozpowszechnianie, przedstawianie i wykonywanie utworu jedynie w celach niekomercyjnych oraz pod warunkiem zachowania go w oryginalnej postaci (nie tworzenia utworów zależnych).



UNIwersYTET ŚLĄSKI  
W KATOWICACH



Biblioteka  
Uniwersytetu Śląskiego



Ministerstwo Nauki  
i Szkolnictwa Wyższego

Anna Bień

## CYCLES IN NIELSEN'S GRAPHS

### Abstract

Nielsen's substitutions are usually used to calculate solutions of equations in free monoids. Each solution can be given as a composition of finitely many such substitutions. The process of solving defines the so called solution graph in which vertices are equations and edges correspond to Nielsen's substitutions. In this paper we deal with solution graphs containing cycles. The main theorem gives some restrictions on the number of variables occurring in equations of such graphs.

### 1. Preliminaries

Let  $M$  be a finite set. The algebra  $(M^*, \Delta, *)$  is called the free monoid over the alphabet  $M$  if  $M^*$  is the set of all finite strings of elements from  $M$ ,  $\Delta$  stands for the empty string, and  $*$  denotes concatenation. Elements of this free monoid are called words and we write  $WU$  instead of  $W * U$ .

Each pair of words  $(U, W)$  is called an equation on words. Every substitution  $\delta$  which satisfies  $\delta(U) = \delta(W)$  is called a solution of the equation  $(U, W)$ .

For every  $x \in M$  and  $W \in M^*$  the symbol  $\delta_{(x,W)}$  denotes the substitution defined by the following formula:  $\delta_{(x,W)} : M \longrightarrow M^*$ ,

$$\delta_{(x,W)}(t) := \begin{cases} W & \text{if } t = x \\ t & \text{if } t \neq x \end{cases}$$

The symbol  $\delta_x$  stands for  $\delta_{(x,\Delta)}$ .

Let  $\delta$  be a substitution and let  $A = (A_L, A_P)$  be an arbitrary equation. The symbol  $\delta(A)$  will denote the reduced form of the  $(\delta(A_L), \delta(A_P))$  in which both words (if they are not empty) begin with different variables.

## 2. Nielsen's graphs

Every equation  $A$  determines at most four Nielsen's substitutions. If  $A = (xU, yW)$ , then four Nielsen's substitutions corresponding to this equation are:  $\delta_x, \delta_y, \delta_{(x,yx)}, \delta_{(y,xy)}$ .

The equation  $A = (xU, \Delta)$  determines only one Nielsen's substitution  $\delta_x$ . Similarly, the equation  $A = (\Delta, yW)$  determines exactly one Nielsen's substitution  $\delta_y$ .

Khmelevski proved [3] that every solution of an equation can be found by use of Nielsen's substitutions. We will consider the so-called solution graph determined by these substitutions. The detailed construction of the solution graph was presented by Badura and Zaionc in [1]. They also showed that every quadric equation has a finite solution tree.

Let  $(W, U)$  be a word equation. The smallest digraph satisfying the conditions

- $(W, U) \in V(G_{(W,U)})$
- if  $A \in V(G_{(W,U)})$  and  $\delta$  is it's Nielsen's substitution, then  $\delta(A) \in V(G_{(W,U)})$
- if  $\{A, \delta(A)\} \subseteq V(G_{(W,U)})$  and  $\delta$  is Nielsen's substitution of  $A$  then  $(A, \delta(A), \delta) \in E(G_{(W,U)})$

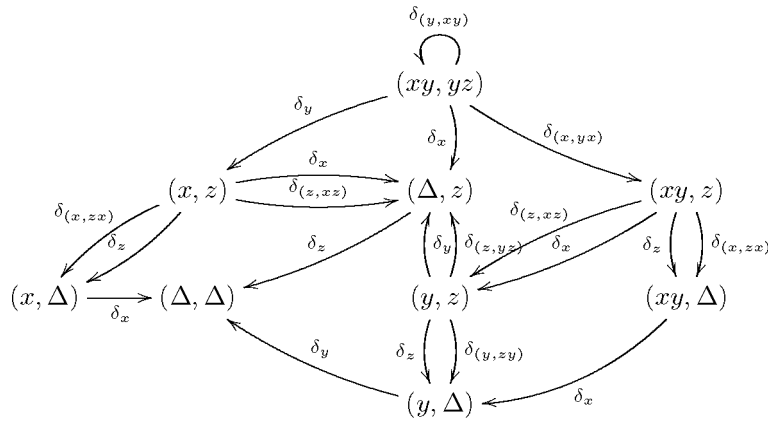
is called Nielsen's graph. Every solution of the equation  $(W, U)$  can be given by a graph path.

In graph theory [2] digraphs are usually defined as pairs  $(V, E)$ , where the vertex set  $V$  is nonempty and the edge set  $E$  consists of pairs of vertices.

In Nielsen's graphs we allow multiple edges, so it is convenient to define an edge as a tuple of three elements  $(A, \delta(A), \delta)$  where  $A$  and  $\delta(A)$  are the vertices and  $\delta$  is the substitution corresponding to the edge.

There are differences in definitions of a path in literature. In this paper we accept that a path is a sequence of edges  $(v_0, v_1, \delta_1), (v_1, v_2, \delta_2) \dots, (v_{n-1}, v_n, \delta_n)$ , where  $n \in \mathbb{N}$ . The sequence does not have to be injective. If  $v_0 = v_n$  then the path is called a cycle.

Following picture gives an example of Nielsen's graph. Every path from the equation  $(xy, yx)$  to the equation of empty words  $(\Delta, \Delta)$  gives a solution of the initial equation.



Nielsen's graph of the equation  $(xy, yx)$

For instance, consider the path  $((xy, yz), (xy, z), \delta_{(x, yx)}), ((xy, z), (y, z), \delta_{(z, xz)}), ((y, z), (\Delta, z), \delta_{(z, yz)}), ((\Delta, z), (\Delta, \Delta), \delta_z)$ . The composition  $\delta := \delta_z \circ \delta_{(z, yz)} \circ \delta_{(z, xz)} \circ \delta_{(x, yx)}$  is a solution given by this path. Indeed, by calculating  $\delta(xy)$  and  $\delta(yz)$  we receive:

$$\delta(xy) = (yxy) = \delta(yz)$$

### 3. Properties of cycles in Nielsen's graphs

Let  $var(A)$  denote the set of all variables occurring in the equation  $A$ . Then the inclusion

$$var(\delta(A)) \subseteq var(A)$$

is satisfied by every Nielsen's substitution  $\delta$  such that  $(A, \delta(A), \delta) \in G_A$ . If  $x \in var(A)$  and  $\delta = \delta_x$ , then  $var(\delta(A)) \neq var(A)$ . This leads to the following conclusion:

LEMMA 1. *If a cycle  $C$  is a subgraph of Nielsen's graph and  $A, B \in V(C)$ , then for every  $x \in M$  we get  $(A, B, \delta_x) \notin E(C)$ .*

PROOF: Let  $A_1, A_2, \dots, A_n$  be all successive vertices of the cycle. Then

$$var(A_n) \subseteq var(A_{n-1}) \subseteq \dots \subseteq var(A_1) \subseteq var(A_n)$$

and hence

$$var(A_n) = var(A_{n-1}) = \dots = var(A_1).$$

Hence, for every two different vertices  $A_i, A_j$  of this cycle and every variable  $x \in M$   $(A_i, A_j, \delta_x) \notin E(C)$ . Otherwise this would lead to a contradiction.  $\square$

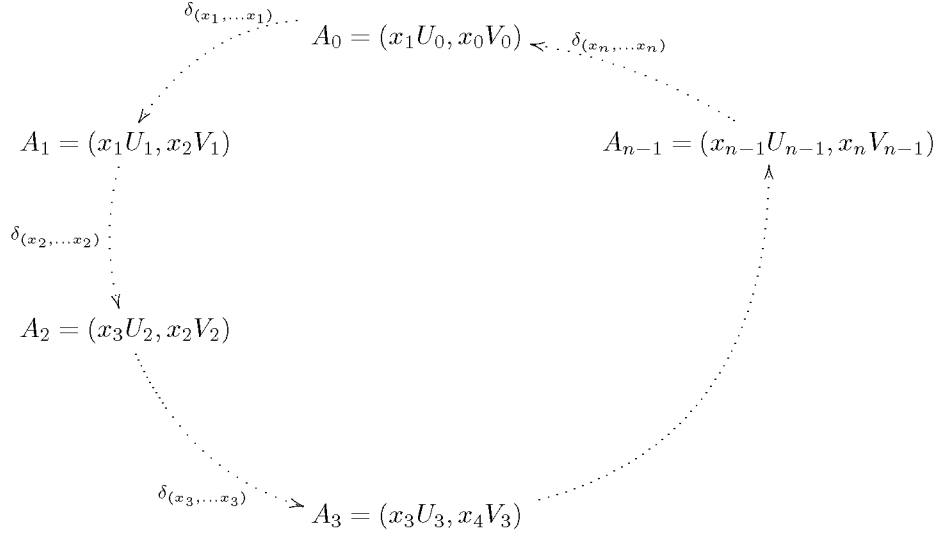
Assume that  $G$  is Nielsen's graph,  $\{A_0, A_1, A_2, \dots, A_{n-1}, A_n\} \subseteq V(G)$  and  $(A_i, A_{i+1}, \delta_{(x, \xi_i x)}) \in E(G)$  for every  $i < n$ . The path  $((A_0, A_1, \delta_{(x, \xi_0 x)}), (A_1, A_2, \delta_{(x, \xi_1 x)}), \dots, (A_{n-1}, A_n, \delta_{(x, \xi_{n-1} x)}))$  will be called *Nielsen's substitution sequence*. It will be denoted

$$A_0 \overset{\delta_{(x, \dots x)}}{\cdots \cdots \cdots} A_n.$$

We will also use the name *main variable* for the variable  $x$ .

From Lemma 1 we know that certain substitutions cannot occur in a cycle.

Hence every cycle in Nielsen's graph has the following form:



We infer that  $x_n = x_0$ . Let  $x$  be an arbitrary variable. The number of occurrences of any variable  $x$  in a word  $A$  is denoted by  $o_x(A)$ .

If  $A$  is an arbitrary equation, then  $o_x(A) = k$  denotes the equations:

$$\begin{cases} o_x(A_L) &= k \\ o_x(A_P) &= k \end{cases}$$

#### 4. Theorem on occurrences of main variables in cycles

We prove a theorem which gives bounds on the number of occurrences of main variables in cycles.

**THEOREM 1.** *If Nielsen's substitution sequence  $A \overset{\delta_{(x, \dots, x)}}{\dots} > B$  in a cycle  $C$  satisfies the condition:  $o_x(A) \neq 0$ , then for every main variable  $x$  of such sequence and for every equation  $B \in V(C)$  we get  $o_x(B) = 1$ .*

LEMMA 2. If  $A = (xU, yW)$ ,  $B = \delta_{(x,yx)}(A)$ , then:

$$\begin{aligned} (1) \quad o_y(B_L) &= o_y(A_L) + o_x(A_L) - 1 \\ (2) \quad o_y(B_P) &= o_y(A_P) + o_x(A_P) - 1 \end{aligned}$$

PROOF OF LEMMA 2: Let  $o_x(A_L) = n, o_x(A_P) = m$ . Hence, there are words  $U_1, U_2 \dots U_n, W_0, W_1, \dots W_m$  such that

$$o_x(U_1) = \dots o_x(U_n) = o_x(W_0) = \dots o_x(W_m) = 0,$$

and

$$A = (xU_1xU_2 \dots xU_n, yW_0xW_1 \dots xW_m).$$

Then

$$B = (xU_1yxU_2 \dots yxU_n, W_0yxW_1 \dots yxW_m).$$

The variable equations result directly from the above form of  $B$ .

Notice that for every variable  $t \neq y$  we get

$$(3) \quad o_t(B) = o_t(A).$$

This lemma can be generalized in the following way:

LEMMA 3. Let  $A_0, A_1, \dots, A_{n-1}, A_n$  be a Nielsen's substitution sequence and let  $x$  be it's main variable. If  $A_i = (xW_i, \xi_i U_i)$  for every  $i \leq n$  and  $k = \text{card}\{i < n : \xi_i = y\}$ , then  $o_y(A_n) = ko_x(A_0) + o_y(A_0) - k$ .

PROOF OF LEMMA 3: Lemma 2 shows the validity of the above equation for  $n = 1$ . Assume that the equation is valid for any Nielsen's substitution sequence of the length  $n$ . Let  $A_0, A_1, \dots, A_n$  be a sequence such that  $A_i = (xW_i, \xi_i U_i)$  for each  $i \leq n$  and  $k = \text{card}\{i < n - 1 : \xi_i = y\}$ . Then  $o_y(A_{n-1}) = ko_x(A_0) + o_y(A_0) - k$ .

If  $\xi_{n-1} = y$ , then  $o_y(A_n) = o_x(A_{n-1}) + o_y(A_{n-1}) - 1$  and  $o_x(A_i) = o_x(A_{i-1})$  for  $0 < i \leq n$ . Hence  $o_y(A_n) = (k+1)o_x(A_0) + o_y(A_0) - (k+1)$ . If  $\xi_{n-1} \neq y$ , then  $o_y(A_n) = o_y(A_{n-1}) = ko_x(A_0) + o_y(A_0) - k$ .

By induction the equation is valid for each  $n$  and each  $k \leq n$ .

PROOF OF THEOREM: Assume that a cycle contains  $n$  maximal Nielsen's substitution sequences, where  $n > 1$ . Let  $A_0, A_1, \dots, A_{n-1}$  be the first

elements of these sequences put in the order as they appear in the cycle and  $x_1, x_2, \dots, x_n$  denote the corresponding main variables.

Knowing that every vertex  $B$  of the cycle belongs to some Nielsen's substitution sequence  $A_{i-1} \overset{\delta(x_i, \dots, x_i)}{\dots} > A_i$ , there exists such  $\beta$  that  $o_{x_1}(B) = \beta o_{x_i}(A_{i-1}) + o_{x_1}(A_{i-1}) - \beta$ . It suffices to show that for any  $i < n$  we have  $o_{x_1}(A_i) = 1$ .

Let  $k \in \{1, \dots, n\}$ . By Lemma 3, for every  $i \in \{1, \dots, n\}$  there exists an integer  $\alpha_i$  such that  $o_{x_k}(A_i) = \alpha_i o_{x_i}(A_{i-1}) + o_{x_k}(A_{i-1}) - \alpha_i$ . If  $x_i$  is the main variable of Nielsen's substitution sequence with the first element  $A_{i-1}$ , then by assumption we get  $o_{x_k}(A_i) \geq o_{x_k}(A_{i-1})$ . Hence  $o_{x_k}(A_0) = o_{x_k}(A_1) = o_{x_k}(A_2) = \dots = o_{x_k}(A_{n-1})$ .

Assume that  $A_0 = (x_1 W_0, x_0 U_0)$ ,  $A_1 = (x_1 W_1, x_2 U_1)$  then we receive  $o_{x_1}(A_0) = 1$ . Indeed, if  $o_{x_0}(A_1) = \alpha_1 o_{x_1}(A_0) + o_{x_0}(A_0) - \alpha_1$  and  $o_{x_0}(A_1) = o_{x_0}(A_0)$  then  $0 = \alpha_1 o_{x_1}(A_0) - \alpha_1$ . Additionally,  $\alpha_1 > 0$  which gives the expected result.  $\square$

If the cycle consists of one sequence of uniform substitutions, then the equation

$$o_{x_0}(A_0) = a o_{x_1}(A_0) + o_{x_0}(A_0) - a$$

is satisfied for some  $a > 0$  and so  $o_{x_1}(A_0) = 1$ . For any vertex  $B$  of the cycle we receive:  $o_{x_1}(B) = o_{x_1}(A_0) = 1$  by (3). Notice that this case does not require any assumption about the main variable. The assumption about the main variable can be also omitted in the case of cycles which contain exactly two maximal Nielsen's substitution sequences. It has not be settled if the assumption is necessary in the general case.

## References

- [1] L. Badura, M. Zaionc, *Parametrizability by regular expressions for equations on words*, **Bulletin of the Section of Logic** 36:1/2 (2007), pp. 79–93.
- [2] J. Bang-Jensen, G. Gutin, **Digraphs: Theory, Algorithms and Applications**, Springer-Verlag, London, 2000.



[3] Yu. I. Khmelevsky, translated by Kandall G.A. from Russian original: *Equation in free semigroups*, **Trudy Mat. Inst. Steklov.** 107(1971), pp. 1–120, in **AMS**, 1976.

Institute of Mathematics, Silesian University,  
Bankowa 14, Katowice 40-007  
Poland  
e-mail: bienann@gmail.com